12 Input-Output Model of Leontief

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- 12.3. Example: 1=transportation sector, 2=vehicle construction

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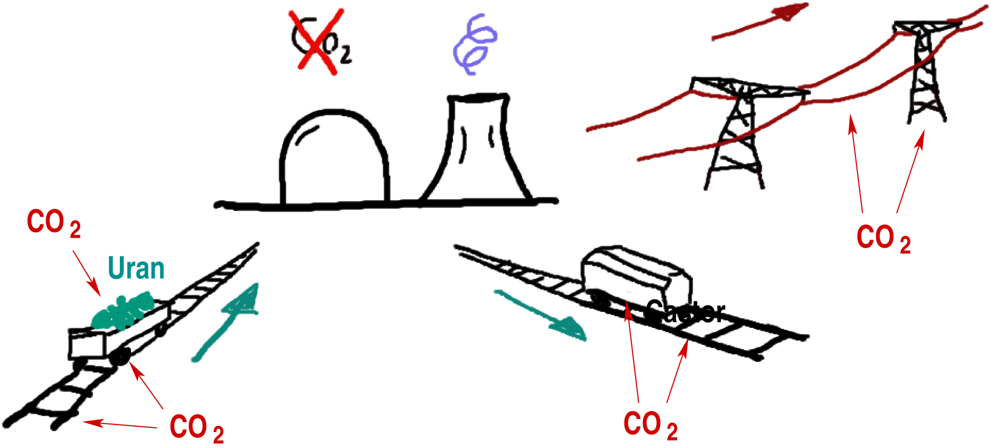

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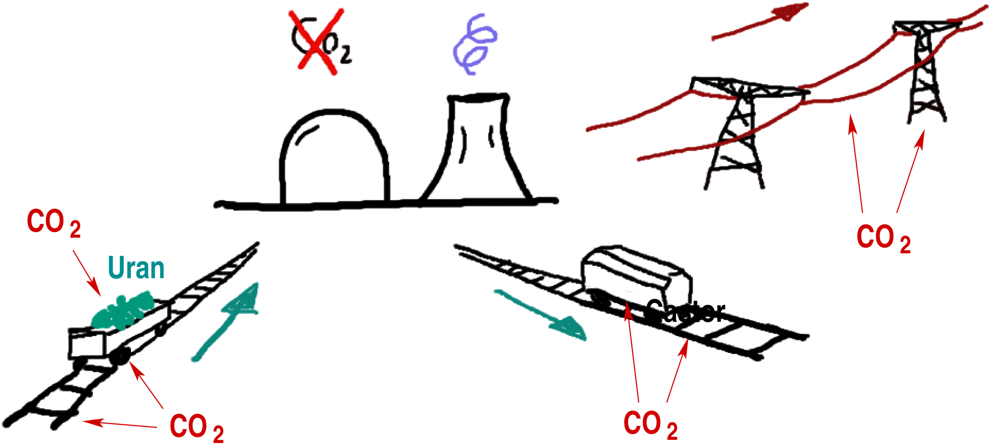

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Problem statement

- In a modern economy, nearly everything is connected to "the rest" of the economy.
- *Wanted:* a quantitative description of the flows of materials, products, services, and information between the different parts of an economy.
- The **input-output model (IOM) of Leontief** tackles this problem by making several assumptions:
  - Every material, product, or service is associated with a certain **sector**
  - To make all flows (kg, €, bytes, ...) commensurable, the common unit is a *monetary unit*, e.g., €
  - The whole system is *linear* and *deterministic*: double input means double output. Particularly, there is no economy of scale
  - The whole system is in the **steady state**, e.g., there are no temporal changes (constant supply and demand); storage (if applicable) is neither built up nor depleted.

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12.2 Specification of the IOM of Leontief

Linear, deterministic coupling of n **sectors** and an **end consumer** in the steady state:

$$x_i =$$

- x_i: Total output of sector i in € or other monetary units per time unit
- y_i: Flow of products/services of sector i to the end consumers (and to sectors that are not explicitely considered)
- x_{ij}: Flow from sector i to j: Sector j needs a supply x_{ij} from sector i to maintain the steady state and to ensure a constant supply y_j to the end consumer
- $A_{ij} = x_{ij}/x_j$: **IO coefficient** reflecting linearity: In order to produce one unit, sector j needs A_{ij} units from all the other sectors i, including the own.

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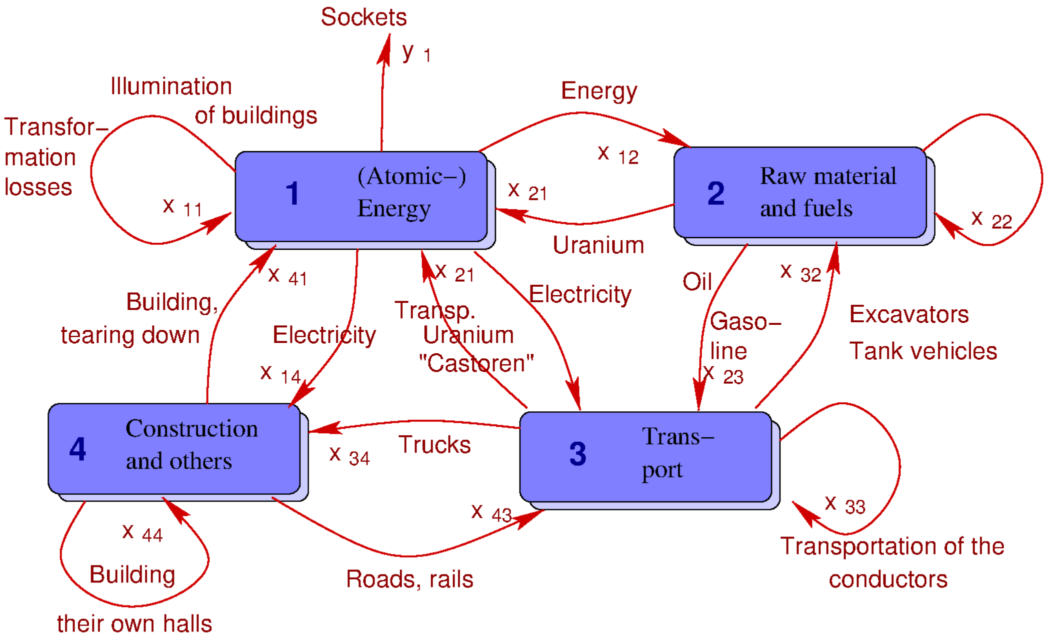
Visualisation of the flows generated by atomic power plants



Total production for a given consumer's supply

IOM equation in vector-matrix notation:

$$\boldsymbol{x} = \mathbf{A} \cdot \boldsymbol{x} + \boldsymbol{y}$$

- $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)'$ **production vector**
- $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)'$ **supply vector**
- $\mathbf{A} = (A_{ij}), i, j = 1 \cdots n$ **IOM coefficient matrix**

Solving for x by writing $(\mathbf{1} - \mathbf{A})x = y$:

$$x = (\mathbf{1} - \mathbf{A})^{-1} y \equiv \mathbf{B}\, y$$

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Meaning of the matrix of the final demand B

- B_{ij} denotes the needed total production from sector i in order to deliver one unit of j to the end consumer (or the not considered sectors) in the steady state
- $\mathbf{B}$ includes all indirect effect *in an infinite recursion* as can be seen from the Taylor expansion:

$$\mathbf{B} = (\mathbf{1} - \mathbf{A})^{-1} = \mathbf{1} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots = \sum_{j=0}^{\infty} \mathbf{A}^j$$

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- $A_{12}A_{21}$: To manage operations, the transport sector must offer aditional transportation for the commutes of the workers/employees of the vehicle making sector (A_{12}), so they can provide additional vehicles (A_{21}) needed by the transportation sector to maintain the steady state.

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- $A_{11}A_{12}A_{21}$: Since also the employees of the transportation companies need transportation (A_{11}), even more transportation supply (A_{12}) must be offered to the employees of the vehicle making companies to get the additionally needed vehicles (A_{21})
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Questions

? Argue that a national economy with sectors i satisfying $\sum_j A_{ij}x_j > x_i$ would not be sustainable or needs external help ("GDR").

! In such an economy, sector i must deliver more units to operate itself $(A_{ii}x_i)$ and the other sectors $(A_{ij}x_j)$ than this sector produces in total (x_i).

? Give reasons why all A_{ij} and B_{ij} are ≥ 0 and $B_{ii} \geq 1$.

! Since sectors *need* products and services from other sectors.

? Assume that the external demand y_k for products/services of sector k suddenly increases by $r_k = 1\%$ (e.g., driven by politics). Give a general expression for the percentaged increase of the GDP in order to re-attain the steady state.

! The change of the demand vector is given by $\Delta y = (0, \ldots r_k y_k, 0, \ldots)'$ and the change of the production vector components by $\Delta x_i = \sum_j B_{ij}y_j = r_k B_{ik} y_k$. Hence, the change of the total GDP is given by $\Delta x = \sum_i \Delta x_i = r_k \sum_i B_{ik} y_k$ and the old GDP itself by $x = \sum_i x_i = \sum_i \sum_j B_{ij} y_j$. Finally, the percentage increase of the total GDP is given by $\Delta x / x$

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! After a sudden change of the demand, the demand vector $\boldsymbol{y}$ is no longer balanced against the available production $(\mathbf{1} - \mathbf{A})\boldsymbol{x}$ and the excess demand or supply is balanced by emptying or filling the stores. If the economy is **demand-driven (Keynes)**, this also induces ramping up/down the production. In the simplest case, the rate of change of the production is proportional to the excess demand,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{\tau_i}\left[y_i(t) - \left((\mathbf{1} - \mathbf{A})\boldsymbol{x}\right)_i\right]$$

where τ_i is the time the sector i needs to adapt to changing demands.

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! In an **economy of scale**, the IO coefficients become smaller with the number of produced units of the target sector which may be modelled, e.g., by

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