10 Inferential statistics II: Tests for Discrete-Choice Models
LR-test and other performance metrics

- 10.1 Significance Tests
- 10.2 Likelihood-Ratio (LR) Test
- 10.3 Goodness-of-Fit Measures

10.1 Significance Tests for Discrete-Choice Models

- The parameter test procedures are exactly the same as that of regression models. Because we only consider the asymptotic limit, the test statistic is always Gaussian:
- Confidence interval of a parameter β_m:

$$\text{CI}_\alpha(\beta_m) = [\hat{\beta}_m - \Delta_\alpha, \hat{\beta}_m + \Delta_\alpha], \quad \Delta_\alpha = z_{1-\alpha/2}\sqrt{V_{mm}}$$

- Test of a parameter β_m for $H_0 : \beta_j = \beta_{j0}, \ \geq \beta_{j0}, \ \text{or} \leq \beta_{j0}$:

$$T = \frac{\hat{\beta}_j - \hat{\beta}_{j0}}{\sqrt{V_{jj}}} \sim N(0,1) \ |H_0^*$$

- p-values for $H_0 : \beta_j = \beta_{j0}, \ \geq \beta_{j0}, \ \text{or} \leq \beta_{j0}$, respectively:

$$p_= = 2\big(1 - \Phi(|t_\text{data}|)\big), \quad p_\leq = 1 - \Phi(t_\text{data}), \quad p_\geq = \Phi(t_\text{data})$$

- As in regression, a factor 4 of more data halves the error

10.2. Likelihood-Ratio (LR) Test

Like in regression (F-test), one sometimes wants to test null hypotheses fixing several parameters *simultaneously* to given values, i.e., H_0 corresponds to a **restraint model**

- H_0: The restraint model with some fixed parameters and M_r remaining parameters describes the data as well as the full model with M parameters
- Test statistics:

$$\lambda^{\text{LR}} = 2\ln\left(\frac{L\left(\hat{\boldsymbol{\beta}}\right)}{L^{\text{r}}\left(\hat{\boldsymbol{\beta}}^{\text{r}}\right)}\right) = 2\left[\tilde{L}\left(\hat{\boldsymbol{\beta}}\right) - \tilde{L}^{\text{r}}\left(\hat{\boldsymbol{\beta}}^{\text{r}}\right)\right] \sim \chi^2(M - M_r) \text{ if } H_0$$

- Data realization: calibrate both M and M_r and evaluate $\lambda^{\text{LR}}_{\text{data}}$
- Result: reject H_0 at α based on the $1-\alpha$ quantile:

$$\lambda^{\text{LR}}_{\text{data}} > \chi^2_{1-\alpha, M-M_r}$$

p-value: $p = 1 - F_{\chi^2(M-M_r)}\left(\lambda^{\text{LR}}_{\text{data}}\right)$

Example: Mode choice for the route to this lecture

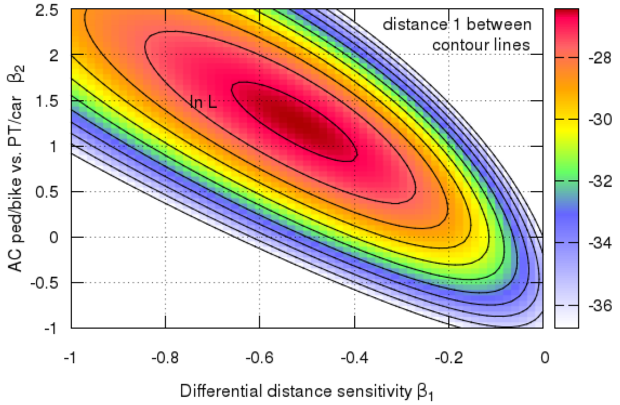
Distance class n	Distance r_n	$i = 1$ (ped/bike)	$i = 2$ (PT/car)
$n = 1$: 0-1 km	0.5 km	7	1
$n = 2$: 1-2 km	1.5 km	6	4
$n = 3$: 2-5 km	3.5 km	6	12
$n = 4$: 5-10 km	7.5 km	1	10
$n = 5$: 10-20 km	15.0 km	0	5

$$
\begin{aligned}
V_{n1}(\beta_1, \beta_2) &= \beta_1 r_n + \beta_2, \\
V_{n2}(\beta_1, \beta_2) &= 0
\end{aligned}
$$

- β_1: Difference in distance sensitivity (utility/km) for choosing ped/bike over PT/car (expected < 0)
- β_2: Utility difference ped/bike over PT/car at zero distance (> 0)

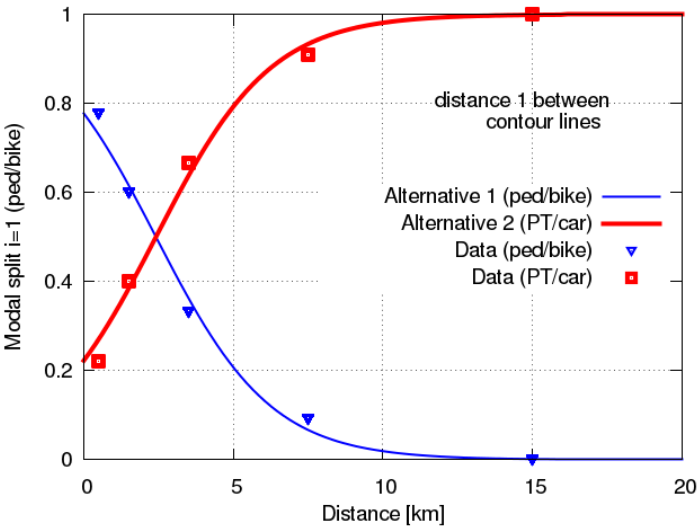
Do the data allow to distinguish this model from the trivial model $V_{ni} = 0$?

LR test for the corresponding Logit models



- H_0: The trivial model $V_{ni} = 0$ describes the data as well as the full model $V_{n1}(\beta_1, \beta_2) = (\beta_1 r_n + \beta_2)\delta_{i1}$
- Test statistics: $\lambda^{\text{LR}} = 2\left[\tilde{L}(\hat{\beta}_1, \hat{\beta}_2) - \tilde{L}(0,0)\right] \sim \chi^2(2)|H_0$
- Data realization (1 $\tilde{L}$-unit per contour): $\lambda^{\text{LR}}_{\text{data}} = 2(-26.5 + 35.5) = 18$
- Decision: Rejection range $\lambda^{\text{LR}} > \chi^2_{2,0.95} = 5.99 \;\Rightarrow$ H_0 rejected.

Fit quality of the full model



- **?** What would be the modeled ped/bike modal split for the null model $V_{ni} = 0$? 50:50
- **?** Read off from the $\tilde{L}$ contour plot the parameter of the AC-only model $V_{ni} = \beta_2 \delta_{i1}$ and give the modeled modal split $\hat{\beta}_2 = \ln(P_1/P_2) = -0.5$, OK with $P_1/P_2 = e^{\hat{\beta}_2} \approx N_1/N_2 = 20/32$
- **?** Motivate the negative correlation between the parameter errors This makes at least sure that, in case of correlated errors, about the same fraction chooses alternative 2 as for the calibrated model

10.3 Goodness-of-Fit Measures

- The parameter tests for equality and the LR test are related to **significance**: Is the more complicated of two nested models significantly better in describing the data?
- This can be used to find the best model using the **top-down ansatz**:

Make is as simple as possible but not simpler!

- Problem: For very big samples, nearly any new parameter gives significance and the top-down ansatz fails
- More importantly: Significance/LR tests cannot give evidence for missing but relevant factors
- A further problem: We cannot compare non-nested models
- Finally, in reality, one often is interested in **effect strength** (difference in the fit and validation quality), not significance

$\Rightarrow$ **we need measures for absolute fit quality**

Information-based goodness-of-fit (GoF) measures

- **Akaike's information criterion**:

$$\mathrm{AIC} = -2\tilde{L} + 2M\frac{N}{N-(M+1)}$$

- **Bayesian information criterion:**

$$\mathrm{BIC} = -2\tilde{L} + M\ln N$$

N: number of decisions; M: number of parameters

- Both criteria give the needed additional information (in bit) to obtain the actual micro-data from the model's prediction, including an over-fitting penalty: the lower, the better.
- Both the AIC and BIC are equivalent to the corresponding GoF measures of regression.
- the BIC focuses more on parsimonious models (low M).
- For nested models satisfying the null hypothesis of the LR test and $N \gg M$, the expected AIC is the same (**verify!**). However, since the AIC is an absolute measure, it allows comparing non-nested models.

GoF measures corresponding to the coefficient of determination R^2 of linear models ($\tilde{L}^0$: log-likelihood of the estimated AC-only or trivial model)

- **LR-Index** resp. **McFadden's R^2**:

$$\rho^2 = 1 - \frac{\tilde{L}}{\tilde{L}^0}$$

- **Adjusted LR-Index/McFadden's R^2:**

$$\bar{\rho}^2 = 1 - \frac{\tilde{L} - M}{\tilde{L}^0}$$

- The LR-Index ρ^2 and the adjusted LR-Index $\bar{\rho}^2$ correspond to the coefficient of determination R^2 and the adjusted coefficient $\tilde{R}^2$ of regression models, respectively: The higher, the better.
- In contrast to regression models, even the best-fitting model has ρ^2 and $\bar{\rho}^2$ values far from 1. Values as low as 0.3 may characterize a good model, *see* *the Example 9.2.1*, while $R^2 = 0.3$ means a really bad fit for a regression model.
- An over-fitted model with M parameters fitting $N = M$ decisions reaches the "ideal" LR-index value $\rho^2 = 1$ while $\bar{\rho}^2$ is near zero.

Questions on GoF metrics

? Discuss the model to be tested, the AC-only model, and the trivial model in the context of weather forecasts
Full forecast info, info from climate table, 50:50

? Give the log-likelihood of the AC-only and trivial models if there are I alternatives and N_i decisions for alternative i (total number of decisions $N = \sum_{i=1}^{I} N_i$)
Trivial model: $P_{ni} = 1/I$, $\tilde{L} = \sum_n \ln P_{i_n} = \sum_i N_i \ln P_i = -N \ln I$;
AC-only model: $P_{ni} = N_i/N$, $\tilde{L} = \sum_i N_i \ln P_i = N \ln N - \sum_i N_i \ln N_i$

? Consider a binary choice situation where the $N/2$ persons with short trips chose the pedestrian/bike option with a probability of 3/4, and the PT/car option with 1/4. The other $N/2$ persons with long trips had the reverse modal split with a ped/bike usage of 25 %, only.
What would be the LR-index for the "perfect" model exactly reproducing the observed 3:1 and 1:3 modal splits for the short and long trips, respectively?
(less than 0.18)