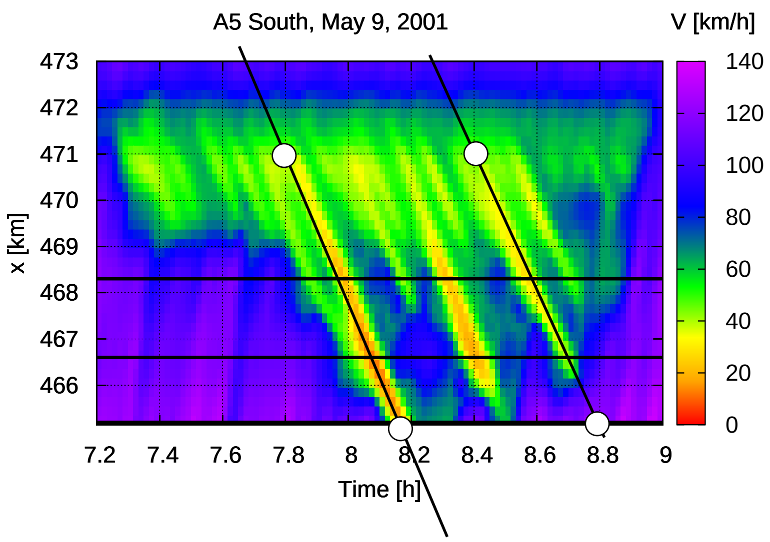
Traffic Flow Dynamics and Simulation

SS 2024, Solutions to Work Sheet 5, page 1

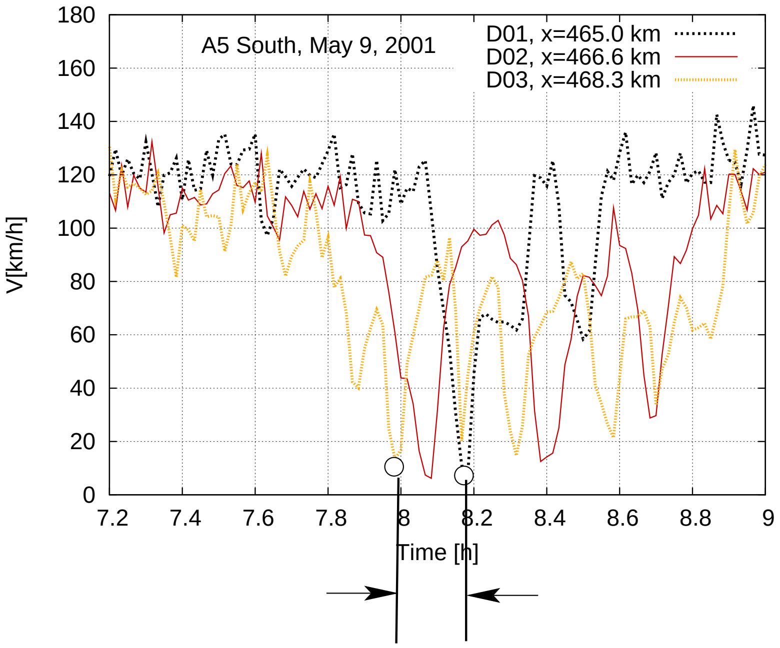
Solution to Problem 5.1: Fundamental diagram as reconstructed from stationary detector data and traffic waves

(a) Propagation velocity w of traffic waves is given by the gradient of the parallel structures (=traffic waves):



$$c = -\frac{6\,\mathrm{km}}{0.38\,\mathrm{h}} = -16\,\mathrm{km/h}.$$

(b) According to the figure below, the most conspicuous traffic wave reaches the upstream boundary of the investigated section (Detekcor D1) at about $t_1 = 8.2\,\text{hours} = 8:12\,\mathrm{h}$. The same wave passed the more downstream detector D3 about 0.2 hours earlier.

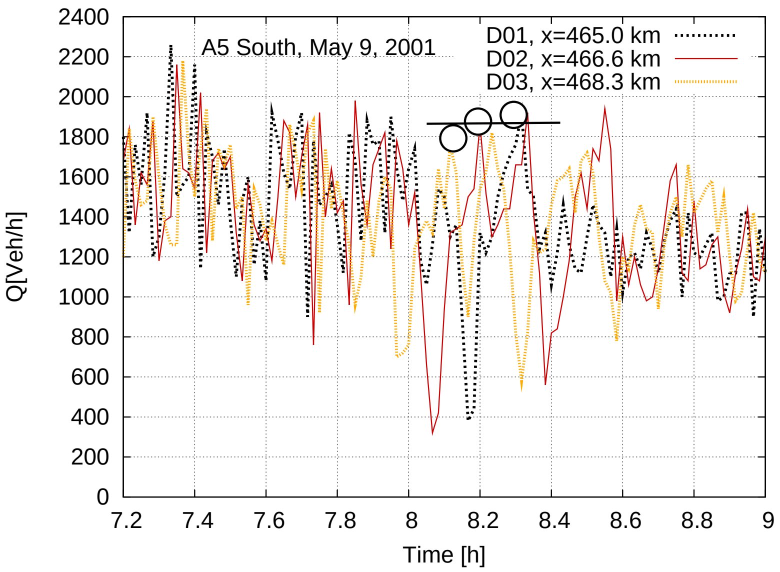


From the distance $x_{13} = 3.3\,\text{km}$ between these detectors, we obtain

$$w = \frac{x_1 - x_3}{t_1 - t_3} = \frac{-3.3\,\text{km}}{0.2\,\text{h}} = -16.5\,\text{km/h}.$$

In view of the uncertainty of reading off precise values from figures, this is consistent with (even remarkably close to) (a).

(c) Based on the speed time series (or the spatiotemporal speed reconstruction) there is essentially free traffic *downstream* the wave considered in the previous question, so the corresponding flow can be used to approximate the (dynamic) road capacity $Q_{\max}$ (the maximum flow *upstream* of the breakdown is somewhat higher, due to the phenomenon of *Capacity drop*). We take the average of the downstrem free-flow regions at the three detectors. Since $w < 0$, this means, we need to read off the flow *after* the wave has passed the respective detector (cf. the figure):



Doing the math results in

$$Q_{\max} \stackrel{!}{=} \frac{1}{3}\left(Q_{\max}^{\mathrm{D1}} + Q_{\max}^{\mathrm{D2}} + Q_{\max}^{\mathrm{D3}}\right) \approx 1\,850\,\mathrm{veh/h}.$$

Notice that the flows must be read *before* demand decreases such that the flow is no longer determined by the capacity but by the demand ($t > 8.7$ hours).

(d) To begin with, the critical density *at capacity* results in equating the flow of the free and congested branches of the tridiagonal fundamental diagram,

$$V_0\rho = \frac{1}{T}\left(1 - \frac{\rho}{\rho_{\max}}\right) \quad \Rightarrow \quad \rho_c = \frac{1}{V_0 T + \frac{1}{\rho_{\max}}} = \frac{1}{V_0 T + l_{\mathrm{eff}}}$$

and the capacity itself by calculating the flow at this density using either branch:

$$Q_{\max} = V_0 \rho_c = \frac{1}{T + \frac{1}{\rho_{\max} V_0}} = \frac{1}{T + \frac{l_{\mathrm{eff}}}{V_0}} \tag{1}$$

(e) The average desired speed V_0 of the drivers is estimated from the periods of (essentially) free traffic before and after the traffic waves:

$$V_0 \approx 120\,\mathrm{km}$$

Notice that the real desired speed presumably is still higher (since also outside the waves there are still obstructing interactions with slower vehicles). However, the triangular fundamental diagram does not include this distinction.

With V_0, w, and $Q_{\max}$, we have three independent unbiased estimates for parameters of the fundamental diagram. With

– The wave speed equal to the gradient of the congested branch,

$$w = Q'_{\text{cong}}(\rho) = -\frac{1}{\rho_{\max} T}$$

and

– The known flow $Q_{\max}$ when evaluating the congested branch at the critical density $\rho_c = Q_{\max}/V_0$,

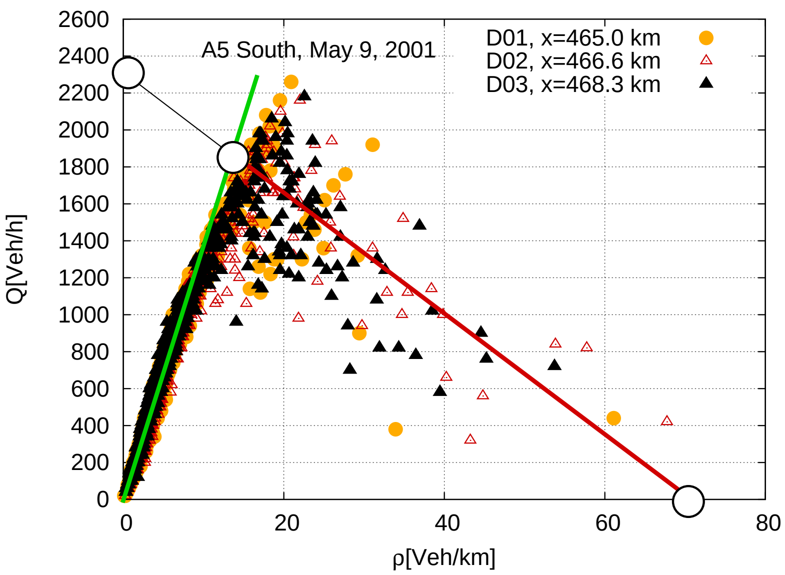
$$Q_{\max} = \frac{1}{T}\left(1 - \frac{\rho_c}{\rho_{\max}}\right),$$

we have two conditions for determining T and $\rho_{\max}$ separately in terms of unbiased estimates,

$$\begin{aligned} T &= \frac{1}{Q_{\max}\left(1 - \frac{w}{V_0}\right)} = 1.7\,\text{s}, \\ \rho_{\max} &= \frac{1}{l_{\text{eff}}} = -\frac{1}{wT} = 130\,\text{veh/km}. \end{aligned}$$

This corresponds to an effective vehicle length (average vehicle length plus average minimum gap) $l_{\text{eff}} = 1/\rho_{\max} = 7.6\,\text{m}$ which is plausible.

(f) Reading off the maximum density and the time-gap parameter T from the linear fit of the congested part of the flow-density scatter plot (cf. the figure below) gives



$$\begin{aligned} \rho_{\max} &= 70\,\text{veh/km}, \\ T &= \frac{1}{2\,300\,1/\text{h}} = 1.57\,\text{s}, \end{aligned}$$

$Q_{\max} = 1\,850\,\text{veh/h}$

as well as the wave velocity

$$w = \frac{-1}{\rho_{\max} T} = -\frac{l_{\text{eff}}}{T} = -9.1\,\text{m/s} = -33\,\text{km/h}$$

As expected,

- The maximum density is grossly underestimated (factor 0.5),
- the absolute value of the wave speed is grossly overestimated (factor 2) ,
- the time gap is moderately underestimated
- the capacity and the density at capacity is unbiased

Discussion

The bias in estimating the space-mean speed from arithmetic time mean speed and the resulting bias in estimating the maximum density is so strong that, in many cases, it makes no sense at all to make a linear fit to the congested flow-density points. In contrast, the systematic bias of the speed estimate does not play a role in estimating w from the passage times of traffic waves at several detectors since only the *argument* of the minimum, not the minimum itself, is relevant for this approach. This argument of the minimum is unique and unbiased as long as the bias is a strictly monotonous (increasing or decreasing) function of the unbiased variable: Here, the biased speed estimate is a strictly monotonously increasing function of the true speed.

Solution to Problem 5.2: Marathon

(a) As for 1d traffic flow, the specific capacity is calculated at the intersection of the free and congested branches of the flow-density relation:

$$V_0\rho = J\left[1 - \frac{\rho}{\rho_{\max}}\right].$$

With $V_0 = \frac{11}{3.6}$ m/s, the critical density “at capacity” is given by

$$\rho_c = \frac{J_0}{V_0 + \frac{J_0}{\rho_{\max}}} = 0.64\,\text{athletes/m}^2$$

and the specific capacity (maximum flow density) itself by

$$J_{\max} = J(\rho_c) = V_0\rho_c = 1.96\,\text{athletes/(s m)}$$

or as function of the runner’s (maximum) speed V_0:

$$J_{\max}(V_0) = \frac{V_0 J_0}{V_0 + \frac{J_0}{\rho_{\max}}}.$$

For groups of fast and slow runners, this gives

$$J_{\max}^{\text{fast}} = 2.08\,\text{runners/(s m)}, \qquad J_{\max}^{\text{slow}}) = 1.81\,\text{runners/(s m)},$$

respectively. The specific capacity only changes little with the runner’s speed. The theoretical asymptotic limit for infinitely fast runners is given by the model parameter $J_0 = 2.5\,\text{runners/(s m)}$. This is in analogy to vehicular traffic where the asymptotoc capacity is given by $1/T$ and the real capacity is always lower.

(b) By definition, the flow is the flow density times the local width $W(x)$ of the course, $Q(x) = JW(x)$. For the $W(8\,\text{km}) = 5\,\text{m}$ wide underpass, this leads (for average-speed runners) to a capacity of

$$Q_{\max} = W J_{\max} = W\ 1.96\,\text{runners/(s m)} = 9.8\,\text{runners/s}$$

(c) The speeds v_i of the individual athletes are drawn from a Gaussian random variable V (expectation μ, variance σ^2), so its density function is given by

$$f(v) = \frac{1}{\sqrt{2\pi\sigma^2}}\ \exp\left[-\frac{(v-\mu)^2}{2\sigma^2}\right].$$

After any given distance x, the speed distribution can be translated into a split-time [German: Zwischenzeit] distribution by transforming the speed v_i into a split time $\tau_i = x/v_i$ (remember that the individual speeds are assumed to be constant over the race). Assuming a mass start, the initially idealized point-like starting field “translates” into a split-time probability density $g(\tau)$ defined by probability conservation:

$$\mathrm{d}p = f(v)\ |\,\mathrm{d}v| = g(\tau)\ |\,\mathrm{d}\tau|$$

(provided $\tau(v) = x/v$ is strictly monotonous which is the case, here). Doing the differential calculation gives

$$g(\tau) = f(v(\tau)) \left| \frac{\mathrm{d}v(\tau)}{\mathrm{d}\tau} \right| = f\left(\frac{v}{\tau}\right) \frac{x}{\tau^2}$$

and, inserting the speed density function $f(.)$:

$$g(\tau) = \frac{x}{\tau^2 \sqrt{2\pi\sigma^2}} \ \exp\left[-\frac{\left(\frac{x}{\tau} - \mu\right)^2}{2\sigma^2} \right]$$

This density function is plotted in Question (d).

(d) The differential probability $\mathrm{d}p$ that an athlet has a split time in $[\tau, \tau + \mathrm{d}\tau]$ is given by $g(\tau)\,\mathrm{d}\tau$. If there are N runners each having an independent realisation from the Gaussian speed variable V, a differential number $\mathrm{d}N = Ng(\tau)\,\mathrm{d}\tau$ of runners passes in the interval $[\tau, \tau + \mathrm{d}\tau]$, hence the flow at location x and at time τ after the start is given by

$$Q(\tau) = \frac{\mathrm{d}N}{\mathrm{d}\tau} = Ng(\tau).$$

In order to prevent congestions, the flow at the maximum of the wave of runners must be below the bottleneck capacity

$$\hat{Q} = N\hat{g} = N \max_{\tau} g(\tau).$$

With $\hat{g} = 1.95\,\mathrm{h}^{-1}$ from the graph, we obtain the critical number of athletes as

$$N_c = \frac{Q_\mathrm{max}}{\hat{g}} = \frac{9.8\,\mathrm{athletes/s}}{\frac{1.95}{3\,600}\,\mathrm{s}^{-1}} \approx 18\,000\,\mathrm{athletes}.$$

(e) By means of the ten-minute delay between the first and the second wave, the flow of runners at the bottleneck becomes more dispersed. The mode of the probability density can be read off the blue curve of the diagram as

$$\hat{g} \approx 1.52\,\mathrm{h}^{-1}$$

(Notice that the blue curves is the unweighted arithmetic average of the red and yellow curves since both waves are assumed to have the same number of participants).

Similarly to Question (d), the maximum number of runners can be estimated as[1]

$$N_c = \frac{Q_\mathrm{max}}{\hat{g}} \approx 23\,000\,\mathrm{athletes}$$

[1] Notice that I did *not* give the calculated precise numbers such as 23 261 feigning a not given precision. Unfortunately, this statistical crime is committed quite often.